

# Integral Equations for ERG

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November 2005

## Abstract

An application of the exact renormalization group equations to the scalar field theory in three dimensional euclidean space is discussed. We show how to modify the original formulation by J. Polchinski in order to find the Wilson-Fisher fixed point using perturbation theory.

## 1 Introduction

The exact renormalization group (ERG) equation, first introduced by K. Wilson [1], was reformulated by J. Polchinski [2] in a form more suitable for perturbation theory. In this paper we apply Polchinski's ERG differential equations to three dimensional scalar field theory.

In the first half of the paper (sects. 2 and 3), we examine the nature of the solutions to the ERG differential equations. We introduce integral equations that combine the differential equations and the asymptotic conditions. The latter specify the solution unambiguously, and the integral equations, under a given set of parameters, have a unique solution. The main issue is self-similarity of the solutions: whether a change of the renormalization scale can be compensated by changing the parameters of the solutions. We show it is impossible to have self-similarity unless we keep an unphysical parameter. In sect. 4, we will modify the ERG differential equations to acquire self-similarity without any unphysical parameter. Finally, in sect. 5, we modify the ERG differential equations further so that the Wilson-Fisher fixed point is accessible by perturbation theory.

Note that in order to have fixed points of ERG, it is necessary to rescale the momenta so that the renormalization momentum scale, say  $\mu$ , remains fixed. Throughout the paper we adopt the convention

$$\mu = 1 \tag{1}$$

## 2 First rewriting: integral equations

The action is given as

$$S(t) = \frac{1}{2} \int_p \phi(p) \phi(-p) \frac{p^2 + m^2 e^{2t}}{K(p)}$$

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$$-\sum_{n=1}^{\infty} \frac{1}{(2n)!} \int_{p_1+\dots+p_{2n}=0} \phi(p_1) \cdots \phi(p_{2n}) \mathcal{V}_{2n}(t; p_1, \dots, p_{2n}) \quad (2)$$

where we use the notation

$$\int_p \equiv \int \frac{d^3 p}{(2\pi)^3}, \quad \int_{p_1+\dots+p_{2n}=0} \equiv \int \prod_{i=1}^{2n} \frac{d^3 p_i}{(2\pi)^3} (2\pi)^3 \delta^{(3)}\left(\sum_{i=1}^{2n} p_i\right) \quad (3)$$

The cutoff function  $K(q)$  is a decreasing positive function of  $q^2$  with the properties

$$K(q) = \begin{cases} 1 & (q^2 < 1) \\ 0 & (q^2 \rightarrow \infty) \end{cases} \quad (4)$$

We also define

$$\Delta(q) \equiv -2q^2 \frac{d}{dq^2} K(q) \quad (5)$$

which vanishes for  $q^2 < 1$  and is positive for  $q^2 > 1$ .

The correlation functions of the scalar field are calculated perturbatively in terms of the propagator

$$\frac{K(p)}{p^2 + m^2 e^{2t}} \quad (6)$$

and the vertices  $\{\mathcal{V}_{2n}(t; p_1, \dots, p_{2n})\}$ . We must introduce specific  $t$ -dependence to the vertices  $\{\mathcal{V}_{2n}(t)\}$  so that<sup>1</sup>

$$\langle \phi(p_1 e^t) \cdots \phi(p_{2n} e^t) \rangle_{m^2 e^{2t}, \mathcal{V}_{2n}(t)} = e^{(y_{2n}-4n)t} \langle \phi(p_1) \cdots \phi(p_{2n}) \rangle_{m^2, \mathcal{V}_{2n}(0)} \quad (7)$$

where

$$y_{2n} \equiv 3 - n \quad (8)$$

is the canonical scale dimension of the  $2n$ -point vertex  $\mathcal{V}_{2n}$ . We note that due to rescaling of momenta under renormalization, the momenta grow as  $e^t$ , and the squared mass grows as  $e^{2t}$ .

The above equality is satisfied if the vertices satisfy the following ERG differential equations, first derived in [2]<sup>2</sup>:

$$\begin{aligned} & \frac{\partial}{\partial t} (e^{-y_{2n}t} \mathcal{V}_{2n}(t; p_1 e^t, \dots, p_{2n} e^t)) \\ &= \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_i e^{-y_{2(k+1)}t} \mathcal{V}_{2(k+1)}(t; p_{i_1} e^t, \dots, p_{i_{2k+1}} e^t, \underbrace{(p_{i_{2(k+1)}} + \dots + p_{i_{2n}}) e^t}_{\equiv p}) \\ & \quad \times \frac{\Delta(p e^t)}{p^2 + m^2} \cdot e^{-y_{2(n-k)}t} \mathcal{V}_{2(n-k)}(t; -p e^t, p_{i_{2(k+1)}} e^t, \dots, p_{i_{2n}} e^t) \\ & \quad + \frac{1}{2} \int_q \frac{\Delta(q e^t)}{q^2 + m^2} e^{-y_{2(n+1)}t} \mathcal{V}_{2(n+1)}(t; q e^t, -q e^t, p_1 e^t, \dots, p_{2n} e^t) \end{aligned} \quad (9)$$

where the index  $i$  runs over the partitions of  $2n$  momenta into two groups.

We will often encounter the right-hand side of the above equation in the rest of this paper. It will save a lot of writing if we introduce a graphical notation. By denoting a vertex  $e^{-y_{2n}t} \mathcal{V}_{2n}(t)$  by

<sup>1</sup>To be precise, this relation is valid only if  $p_i^2 < e^{-2t}$  for all  $i$ .

<sup>2</sup> $e^{-y_{2n}t} \mathcal{V}_{2n}(t; p_1 e^t, \dots)$  should be replaced by  $\mathcal{V}_{2n}(t; p_1, \dots)$ , if we do not rescale the momenta under renormalization.



and  $\frac{\Delta(pe^t)}{p^2+m^2}$  by a thick line



we can rewrite the ERG differential equations as

$$\frac{\partial}{\partial t} \text{ (circle with two external lines) } = \sum_{\text{partitions}} \text{ (circle with two external lines) } \text{---} \text{ (circle with two external lines) } + \frac{1}{2} \int_q \text{ (circle with two external lines and a red loop) }$$

The solutions of the ERG differential equations that originate from the trivial UV fixed point at  $t = -\infty$  can be characterized completely by their asymptotic behaviors as  $t \rightarrow -\infty$ :

$$e^{-2t} \mathcal{V}_2(t; pe^t, -pe^t) \xrightarrow{t \rightarrow -\infty} \lambda e^{-t} a_2 + \lambda^2 (-Ct + b_2) \quad (10)$$

$$e^{-t} \mathcal{V}_4(t; p_1 e^t, \dots, p_4 e^t) \xrightarrow{t \rightarrow -\infty} -\lambda \quad (11)$$

$$e^{-y_{2n} t} \mathcal{V}_{2n \geq 6}(t; p_1 e^t, \dots, p_{2n} e^t) \xrightarrow{t \rightarrow -\infty} 0 \quad (12)$$

where the two constants  $\lambda$ ,  $b_2$  are arbitrary, but  $a_2$  and  $C$  are determined uniquely. The constant  $\lambda > 0$  is of course the coupling constant, while  $b_2$  only shifts the origin of the logarithmic parameter  $t$  and is expected to be unphysical. Hence, the vertices  $\mathcal{V}_{2n}(t; p_1, \dots, p_{2n})$  depend on three arbitrary parameters<sup>3</sup>:

1.  $m^2$  which appears in the ERG differential equations
2.  $\lambda > 0$  which determines the asymptotic behavior of  $\mathcal{V}_4$
3.  $b_2$  which determines the asymptotic behavior of  $\mathcal{V}_2$

When we wish to show the parametric dependence explicitly, we will use the following notation

$$\mathcal{V}_{2n}(t; p_1, \dots, p_{2n}; m^2, \lambda, b_2)$$

Before introducing integral equations, let us make a simple observation. If the vertices  $\{\mathcal{V}_{2n}(t; p_1, \dots, p_{2n})\}$  solve the ERG differential equations for the squared mass  $m^2$ , it is straightforward to show that the shifted vertices  $\{\mathcal{V}_{2n}(t + \Delta t; p_1, \dots, p_{2n})\}$  also satisfy the ERG differential equations for the squared mass  $m^2 e^{2\Delta t}$ . Examining the asymptotic behaviors of the shifted vertices, we conclude

$$\begin{aligned} & \mathcal{V}_{2n}(t + \Delta t; p_1, \dots, p_{2n}; m^2, \lambda, b_2) \\ = & \mathcal{V}_{2n}(t; p_1, \dots, p_{2n}; m^2 e^{2\Delta t}, \lambda e^{\Delta t}, b_2 - C\Delta t) \end{aligned} \quad (13)$$

We call this property **self-similarity** following the standard nomenclature, meaning that a shift of the logarithmic scale variable  $t$  can be absorbed by changes of the parameters of the vertices. (Fig. 1.) The above shows that it is essential to keep  $b_2$  if we wish to have self-similarity. However, as we will show more explicitly in the next section,  $b_2$  is an unphysical parameter. If we wish to

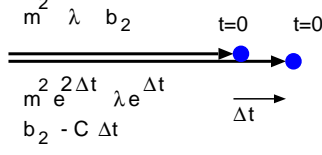


Figure 1: The ERG trajectory specified by  $m^2 e^{2\Delta t}$ ,  $\lambda e^{\Delta t}$ ,  $b_2 - C\Delta t$  is the same as the one specified by  $m^2$ ,  $\lambda$ ,  $b_2$ . But the parameter  $t$  is shifted by  $\Delta t$ .

have self-similarity with only two parameters  $m^2$  and  $\lambda$ , we need to modify the ERG differential equations themselves. This will be discussed in sect.4.

Now, to compute the vertices perturbatively, it is convenient to convert the ERG **differential** equations into **integral** equations that incorporate the asymptotic behaviors explicitly. The integral equations of this type have been discussed extensively for the four dimensional scalar theory in ref. [3], and we merely transpose the results to three dimensions. For the two-point vertex we obtain

$$\begin{aligned}
& e^{-2t} \mathcal{V}_2(t; pe^t, -pe^t) \\
= & \int_{-\infty}^t dt' \left[ e^{-2t'} \mathcal{V}_2(t'; pe^{t'}, -pe^{t'}) \frac{\Delta(pe^{t'})}{p^2 + m^2} e^{-2t'} \mathcal{V}_2(t'; pe^{t'}, -pe^{t'}) \right. \\
& \left. + \frac{1}{2} \int_q \frac{\Delta(qe^{t'})}{q^2 + m^2} e^{-t'} \mathcal{V}_4(t'; qe^{t'}, -qe^{t'}, pe^t, -pe^t) + e^{-t'} \lambda a_2 + \lambda^2 C \right] \\
& + e^{-t} \lambda a_2 + \lambda^2 (-Ct + b_2)
\end{aligned} \tag{14}$$

The integral over  $t'$  is convergent thanks to the UV subtraction. To compensate for the unwanted  $t$  dependence of the subtraction, we must introduce finite counterterms. The parameter  $b_2$  enters as an integration constant. For the four-point vertex, we obtain

$$\begin{aligned}
& e^{-t} \mathcal{V}_4(t; p_1 e^t, \dots, p_4 e^t) \\
= & \int_{-\infty}^t dt' \left[ \sum_{i=1}^4 e^{-2t'} \mathcal{V}_2(t'; p_i e^{t'}, -p_i e^{t'}) \frac{\Delta(p_i e^{t'})}{p_i^2 + m^2} \cdot e^{-t'} \mathcal{V}_4(t'; p_1 e^{t'}, \dots, p_4 e^{t'}) \right. \\
& \left. + \frac{1}{2} \int_q \frac{\Delta(qe^{t'})}{q^2 + m^2} \mathcal{V}_6(t'; qe^{t'}, -qe^{t'}, p_1 e^{t'}, \dots, p_4 e^{t'}) \right] - \lambda
\end{aligned} \tag{15}$$

The integral over  $t'$  is convergent. The coupling  $\lambda$  is introduced as an integration constant. For the six-point and higher vertices, we obtain

$$e^{-y_{2n} t} \mathcal{V}_{2n \geq 6}(t; p_1 e^t, \dots, p_{2n} e^t)$$

<sup>3</sup>We fix the choice of the cutoff function  $K$ .

$$\begin{aligned}
&= \int_{-\infty}^t dt' \left[ \sum_{\text{partitions}} \text{Diagram} + \frac{1}{2} \int_q \frac{\Delta(qe^{t'})}{q^2 + m^2} e^{-y_{2(n+1)}t'} \mathcal{V}_{2(n+1)}(t'; qe^{t'}, -qe^{t'}, p_1 e^{t'}, \dots, p_{2n} e^{t'}) \right] \quad (16)
\end{aligned}$$

No integration constant is necessary. In the above integral equations, the convergence of the  $t'$  integral guarantees the expected asymptotic behaviors.

Solving the integral equations recursively in powers of  $\lambda$ , we obtain

$$a_2 = \frac{1}{2} \int_q \frac{\Delta(q)}{q^2} \quad (17)$$

$$C = -\frac{1}{2} \int_q \frac{\Delta(q)}{q^2} \int_r \frac{1-K(r)}{r^2} \frac{1-K(q+r)}{(q+r)^2} = -\frac{1}{(4\pi)^2} \frac{1}{6} \quad (18)$$

While  $a_2$  depends on the choice of the cutoff function  $K$ , the constant  $C$  is independent.

### 3 Change of field variables

In modifying ERG differential equations, we use linear changes of field variables as the main tool. This has been discussed in detail for the four dimensional theory in ref. [4].

#### 3.1 First type

We introduce the following infinitesimal change of field variables:

$$\phi(p) \rightarrow \phi(p) \left( 1 + \frac{1}{2} s(p) \right) \quad (19)$$

where  $s(p)$  is given by

$$s(p) \equiv -\delta z + (1 - K(p)) \left( \delta z + \frac{\delta m^2}{p^2 + m^2 e^{2t}} \right) \quad (20)$$

Both  $\delta z$  and  $\delta m^2$  are infinitesimal constants. Under this change of variables, the squared mass changes as

$$m^2 \rightarrow m^2 + \delta m^2 \quad (21)$$

and the vertices change as

$$\begin{aligned}
\delta \mathcal{V}_2(t; p, -p) &= \delta z(p^2 + m^2 e^{2t}) + \delta m^2 \\
&\quad + (1 + s(p)) \mathcal{V}_2(t; p, -p) \quad (22)
\end{aligned}$$

$$\delta \mathcal{V}_{2n \geq 4}(t; p_1, \dots, p_{2n}) = \frac{1}{2} \sum_{i=1}^{2n} s(p_i) \cdot \mathcal{V}_{2n}(t; p_1, \dots, p_{2n}) \quad (23)$$

Obviously,  $\delta m^2$  and  $\delta z$  are the counterterms for the squared mass and wave function, respectively. It is this type of change of variables that we will use in sects. 4&5 to modify the ERG differential equations.

### 3.2 Second type

With  $u(p)$  as an arbitrary infinitesimal function of  $p^2$ , we introduce the change of variables:

$$\phi(p) \rightarrow \phi(p) \left(1 + \frac{1}{2}u(p)\right) \quad (24)$$

This changes the propagator as

$$\frac{K(p)}{p^2 + m^2 e^{2t}} \rightarrow \frac{K(p)}{p^2 + m^2 e^{2t}} (1 - u(p)) \quad (25)$$

and the vertices as

$$\mathcal{V}_{2n}(t; p_1, \dots, p_{2n}) \rightarrow \left(1 + \frac{1}{2} \sum_{i=1}^{2n} u(p_i)\right) \mathcal{V}_{2n}(t; p_1, \dots, p_{2n}) \quad (26)$$

If we assume that  $u(p)$  is local, i.e.,

$$u(p) = 0 \quad \text{if} \quad p^2 < 1 \quad (27)$$

then we can absorb the change of the propagator by changing the vertices as follows:

$$\begin{aligned} \delta \mathcal{V}_{2n}(t; p_1, \dots, p_{2n}) &= \frac{1}{2} \sum_{i=1}^{2n} u(p_i) \cdot \mathcal{V}_{2n}(t; p_1, \dots, p_{2n}) \\ &\quad - \sum_{\text{partitions}} \left( \text{diagram of a circle with } n \text{ external lines} \right) \frac{K(p)u(p)}{p^2 + m^2 e^{2t}} \left( \text{diagram of a circle with } n \text{ external lines} \right) \\ &\quad - \frac{1}{2} \int_q \frac{K(q)u(q)}{q^2 + m^2 e^{2t}} \mathcal{V}_{2(n+1)}(t; q, -q, p_1, \dots, p_{2n}) \end{aligned} \quad (28)$$

The most general linear change of field variables is obtained by combining the first and second types.

### 3.3 Unphysical nature of the parameter $b_2$

Now we are ready to discuss the unphysical nature of the parameter  $b_2$  in detail. We combine the first and second types with the choice

$$\delta m^2 = \epsilon \lambda^2 e^{2t} \quad (29)$$

$$\delta z = 0 \quad (30)$$

$$u(pe^t) = u(t; p) \equiv (1 - K(pe^t)) \frac{\epsilon \lambda^2}{p^2 + m^2} \quad (31)$$

so that

$$s(pe^t) = u(t; p) \quad (32)$$

Then, we obtain the following infinitesimal change of vertices:

$$\begin{aligned} e^{-2t} \delta \mathcal{V}_2(t; pe^t, -pe^t) &= \epsilon \lambda^2 + 2u(t; p) \cdot e^{-2t} \mathcal{V}_2(t; pe^t, -pe^t) \\ &\quad - e^{-2t} \mathcal{V}_2(t; pe^t, -pe^t) \frac{K(pe^t)u(t; p)}{p^2 + m^2} e^{-2t} \mathcal{V}_2(t; pe^t, -pe^t) \\ &\quad - \frac{1}{2} \int_q \frac{K(qe^t)u(t; q)}{q^2 + m^2} e^{-t} \mathcal{V}_4(t; qe^t, -qe^t, pe^t, -pe^t) \end{aligned} \quad (33)$$

and

$$\begin{aligned}
e^{-y_{2n}t} \delta \mathcal{V}_{2n \geq 4}(t; p_1 e^t, \dots, p_{2n} e^t) &= \sum_{i=1}^{2n} u(t; p_i) \cdot e^{-y_{2n}t} \mathcal{V}_{2n}(t; p_1 e^t, \dots, p_{2n} e^t) \\
&- \sum_{\text{partitions}} \text{Diagram} \cdot \frac{K(pe^t)u(t; p)}{p^2 + m^2} \cdot \text{Diagram} \\
&- \frac{1}{2} \int_q \frac{K(qe^t)u(t; q)}{q^2 + m^2} e^{-y_{2(n+1)}t} \mathcal{V}_{2(n+1)}(t; qe^t, -qe^t, p_1 e^t, \dots, p_{2n} e^t) \quad (34)
\end{aligned}$$

The above change of variables is very special in the sense that the modified vertices  $\{(\mathcal{V}_{2n} + \delta \mathcal{V}_{2n})(t)\}$  satisfy the same ERG differential equations as  $\{\mathcal{V}_{2n}(t)\}$  except that the squared mass parameter  $m^2$  is replaced by

$$m^2 + \delta m^2 = m^2 + \epsilon \lambda^2 \quad (35)$$

It is straightforward (but tedious) to check this.

Since the vertices  $\{(\mathcal{V}_{2n} + \delta \mathcal{V}_{2n})(t)\}$  are obtained from  $\{\mathcal{V}_{2n}(t)\}$  by the change of field variables

$$\phi(pe^t) \longrightarrow \phi(pe^t) (1 + u(t; p)) \quad (36)$$

the correlation functions do not change <sup>4</sup>:

$$\langle \phi(p_1) \cdots \phi(p_{2n}) \rangle_{m^2 e^{2t}; \mathcal{V}(t)} = \langle \phi(p_1) \cdots \phi(p_{2n}) \rangle_{(m^2 + \epsilon \lambda^2) e^{2t}; (\mathcal{V} + \delta \mathcal{V})(t)} \quad (37)$$

Examining the change of the parameters  $\lambda$  and  $b_2$  under the above infinitesimal change

$$e^{-2t} \delta \mathcal{V}_2(t; pe^t, -pe^t) \xrightarrow{t \rightarrow -\infty} \epsilon \lambda^2 \quad (38)$$

$$e^{-t} \delta \mathcal{V}_4(t; p_1 e^t, \dots, p_4 e^t) \xrightarrow{t \rightarrow -\infty} 0 \quad (39)$$

we obtain

$$\delta b_2 = \epsilon, \quad \delta \lambda = 0 \quad (40)$$

Hence, we find

$$\begin{aligned}
&(\mathcal{V}_{2n} + \delta \mathcal{V}_{2n})(t; p_1, \dots, p_{2n}; m^2, \lambda, b_2) \\
&= \mathcal{V}_{2n}(t; p_1, \dots, p_{2n}; m^2 + \epsilon \lambda^2, \lambda, b_2 + \epsilon) \quad (41)
\end{aligned}$$

Therefore, the theory parametrized by  $m^2, \lambda, b_2$  gives the same correlation functions as the theory with  $m^2 + \epsilon \lambda^2, \lambda, b_2 + \epsilon$ :

$$(m^2, \lambda, b_2) \xLeftrightarrow{\text{equivalent}} (m^2 + \epsilon \lambda^2, \lambda, b_2 + \epsilon) \quad (42)$$

This shows the unphysical nature of the parameter  $b_2$ . For instance, we can adopt the convention

$$b_2 = 0 \quad (43)$$

since, given an arbitrary ERG trajectory with  $b_2 \neq 0$ , we can always find an equivalent ERG trajectory satisfying this condition.

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<sup>4</sup>Strictly speaking, we must restrict  $p_i^2 < 1$  for all  $i$ .

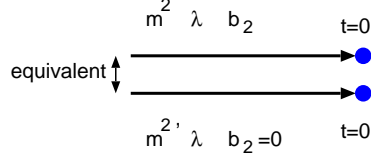


Figure 2: Given a trajectory with  $b_2 \neq 0$ , we can find an equivalent trajectory satisfying  $b_2 = 0$ . ( $m^{2'} = m^2 - b_2 \lambda^2$ )

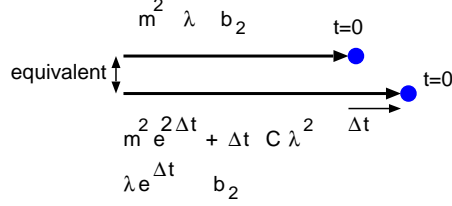


Figure 3: The ERG trajectory specified by  $m^2, \lambda, b_2$  is equivalent with the trajectory specified by  $m^2 e^{2\Delta t} + \Delta t \cdot C \lambda^2, \lambda e^{\Delta t}, b_2$ . But the logarithmic scale parameter  $t$  is shifted by  $\Delta t$ .

### 3.4 RG equations from ERG

Using the convention  $b_2 = 0$ , the ERG trajectories are now specified only by  $m^2$  and  $\lambda$ . We wish to derive the RG equations for  $m^2$  and  $\lambda$  from ERG.

Let us recall the result (13). Using this, we can shift the logarithmic scale parameter  $t$  by an infinitesimal  $\Delta t$  in the equivalence (42):

$$(m^2 e^{2\Delta t}, \lambda e^{\Delta t}, b_2 - C \Delta t) \xrightarrow{\text{equivalent}} (m^2 e^{2\Delta t} + \Delta t \cdot C \lambda^2, \lambda e^{\Delta t}, b_2) \quad (44)$$

where we have chosen  $\epsilon = C \Delta t$ . This implies that the ERG trajectory specified by  $m^2, \lambda, b_2$  gives the same correlation functions as the trajectory specified by  $m^2 e^{2\Delta t} + \Delta t \cdot C \lambda^2, \lambda e^{\Delta t}, b_2$ , except that the logarithmic scale parameter  $t$  of the latter trajectory is shifted by  $\Delta t$ . Thus, we obtain

$$\begin{aligned} & \langle \phi(p_1 e^{\Delta t}) \cdots \phi(p_{2n} e^{\Delta t}) \rangle_{(m^2 e^{2\Delta t} + \Delta t \cdot C \lambda^2) e^{2t}; \mathcal{V}(t; m^2 e^{2\Delta t} + \Delta t \cdot C \lambda^2, \lambda e^{\Delta t}, b_2)} \\ &= e^{(y_{2n} - 4n) \Delta t} \langle \phi(p_1) \cdots \phi(p_{2n}) \rangle_{m^2 e^{2t}; \mathcal{V}(t; m^2, \lambda, b_2)} \end{aligned} \quad (45)$$

Taking  $b_2 = 0$  and  $t = 0$ , we obtain

$$\begin{aligned} & \langle \phi(p_1 e^{\Delta t}) \cdots \phi(p_{2n} e^{\Delta t}) \rangle_{m^2 e^{2\Delta t} + \Delta t \cdot C \lambda^2; \mathcal{V}(t=0; m^2 e^{2\Delta t} + \Delta t \cdot C \lambda^2, \lambda e^{\Delta t}, 0)} \\ &= e^{(y_{2n} - 4n) \Delta t} \langle \phi(p_1) \cdots \phi(p_{2n}) \rangle_{m^2; \mathcal{V}(t=0; m^2, \lambda, 0)} \end{aligned} \quad (46)$$

This is the standard RG equation for the  $\phi^4$  theory in three dimensions. The RG equations for the parameters  $m^2, \lambda$  are given by

$$\frac{dm^2}{dt} = 2m^2 + C \lambda^2 \quad (47)$$

$$\frac{d\lambda}{dt} = \lambda \quad (48)$$



What is counterintuitive about the above result is that the ordinary RG equations are obtained by comparing two **different** ERG trajectories.

## 4 Second rewriting: self-similarity

We have explained one undesirable feature of the ERG differential equations: for self-similarity we must keep an unphysical parameter  $b_2$  in addition to the physical parameters  $m^2, \lambda$ . The purpose of this section is to modify the ERG differential equations so that the solutions, parametrized only by two physical parameters, are self-similar.

We modify the ERG differential equations so that the asymptotic behavior of the two-point vertex is given by

$$e^{-2t}\mathcal{V}_2(t; pe^t, -pe^t) \xrightarrow{t \rightarrow -\infty} \lambda e^{-t} a_2 \quad (49)$$

where  $a_2$  is given by (17), without any terms of order 1 proportional to  $\lambda^2$ . We modify the ERG differential equations by adding a mass counterterm as discussed in the previous section. The modified ERG differential equations are given as follows:

$$\begin{aligned} & \frac{\partial}{\partial t} (e^{-2t}\mathcal{V}_2(t; pe^t, -pe^t)) = -a_2\lambda(t)^2 e^{-2t} \\ & + s(pe^t; m^2(t), \lambda(t)) \cdot e^{-2t}\mathcal{V}_2(t; pe^t, -pe^t) \\ & + e^{-2t}\mathcal{V}_2(t; pe^t, -pe^t) \frac{\Delta(pe^t)}{p^2 + e^{-2t}m^2(t)} e^{-2t}\mathcal{V}_2(t; pe^t, -pe^t) \\ & + \frac{1}{2} \int_q \frac{\Delta(qe^t)}{q^2 + e^{-2t}m^2(t)} e^{-t}\mathcal{V}_4(t; qe^t, -qe^t, pe^t, -pe^t), \quad (50) \\ & \frac{\partial}{\partial t} (e^{-y_{2n}t}\mathcal{V}_{2n \geq 4}(t; p_1 e^t, \dots, p_{2n} e^t)) \\ & = \frac{1}{2} \sum_{i=1}^{2n} s(p_i e^t; m^2(t), \lambda(t)) \cdot e^{-y_{2n}t}\mathcal{V}_{2n}(t; p_1 e^t, \dots, p_{2n} e^t) \\ & + \sum_{\text{partitions}} \left( \text{Diagram 1} \right) \frac{\Delta(pe^t)}{p^2 + e^{-2t}m^2(t)} \left( \text{Diagram 2} \right) \\ & + \frac{1}{2} \int_q \frac{\Delta(qe^t)}{q^2 + e^{-2t}m^2(t)} e^{-y_{2(n+1)}t}\mathcal{V}_{2(n+1)}(t; qe^t, -qe^t, p_1 e^t, \dots, p_{2n} e^t) \quad (51) \end{aligned}$$

where the running parameters are given by

$$m^2(t) \equiv e^{2t} (m^2 + C\lambda^2 t) \quad (52)$$

$$\lambda(t) \equiv e^t \lambda \quad (53)$$

and

$$s(p; m^2, \lambda) \equiv C\lambda^2 \frac{1 - K(p)}{p^2 + m^2} \quad (54)$$

As in the case of the original ERG differential equations, the solutions originating from the trivial fixed point at  $t = -\infty$  are completely characterized by

the asymptotic behaviors, which are given in this case as follows:

$$e^{-2t}\mathcal{V}_2(t; pe^t, -pe^t) \xrightarrow{t \rightarrow -\infty} e^{-t}\lambda a_2 \quad (55)$$

$$e^{-t}\mathcal{V}_4(t; p_1 e^t, \dots, p_4 e^t) \xrightarrow{t \rightarrow -\infty} -\lambda \quad (56)$$

$$e^{-y_{2n}t}\mathcal{V}_{2n \geq 4}(t; p_1 e^t, \dots, p_{2n} e^t) \xrightarrow{t \rightarrow -\infty} 0 \quad (57)$$

We can construct integral equations that incorporate the above asymptotic behaviors. For the two-point vertex we obtain

$$\begin{aligned} & e^{-2t}\mathcal{V}_2(t; pe^t, -pe^t) \\ &= \int_{-\infty}^t dt' \left[ e^{-2t'}\mathcal{V}_2(t'; pe^{t'}, -pe^{t'}) \frac{\Delta(pe^{t'})}{p^2 + e^{-2t'}m^2(t')} e^{-2t'}\mathcal{V}_2(t'; pe^{t'}, -pe^{t'}) \right. \\ & \quad + \frac{1}{2} \int_q \frac{\Delta(qe^{t'})}{q^2 + e^{-2t'}m^2(t')} e^{-t'}\mathcal{V}_4(t'; qe^{t'}, -qe^{t'}, pe^{t'}, -pe^{t'}) \\ & \quad \left. + e^{-t'}\lambda a_2 + C\lambda^2 + s(pe^{t'}; m^2(t'), \lambda(t')) e^{-2t'}\mathcal{V}_2(t'; pe^{t'}, -pe^{t'}) \right] \\ & \quad + e^{-t}\lambda a_2 \end{aligned} \quad (58)$$

For the four- and higher-point vertices, we obtain

$$\begin{aligned} & e^{-y_{2n}t}\mathcal{V}_{2n \geq 4}(t; p_1 e^t, \dots, p_{2n} e^t) \\ &= \int_{-\infty}^t dt' \left[ \sum_{\text{partitions}} \text{diagram} + \frac{1}{2} \int_q \text{diagram} \right. \\ & \quad \left. + \frac{1}{2} \sum_{i=1}^{2n} s(p_i e^{t'}; m^2(t'), \lambda(t')) \cdot e^{-y_{2n}t'}\mathcal{V}_{2n}(t'; p_1 e^{t'}, \dots, p_{2n} e^{t'}) \right] - \lambda \delta_{n,2} \end{aligned} \quad (59)$$

where the thick line with momentum  $q$  denotes

$$\frac{\Delta(qe^{t'})}{q^2 + e^{-2t'}m^2(t')}$$

By shifting the integration variable  $t'$  by  $t$  so that the range of integration becomes  $[-\infty, 0]$ , we find that the vertices are indeed self-similar:

$$\boxed{\mathcal{V}_{2n}(t; p_1, \dots, p_{2n}) = F_{2n}(p_1, \dots, p_{2n}; m^2(t), \lambda(t))} \quad (60)$$

where  $F_{2n}$  has no explicit  $t$  dependence.

Thus, we have accomplished the goal of this section, and the ERG flows now coincide with the RG flows of the running parameters  $m^2(t)$  and  $\lambda(t)$ . The RG equations are the same as those derived at the end of the previous section, and they are identical to the RG equations for the dimensionally regularized theory with the minimal subtraction.

However, there is a problem with the above RG equations: the Wilson-Fisher fixed point is hidden at infinite  $\lambda$ . Let us define an RG invariant

$$R(m^2, \lambda) \equiv \frac{m^2}{\lambda^2} - C \ln \lambda \quad (61)$$

which takes a critical value, say  $R_{cr}$ , for the massless theory. Then, the Wilson-Fisher fixed point lies at the infinite  $\lambda$  limit of the critical RG trajectory  $R = R_{cr}$ . In order to obtain the fixed point at finite values of parameters, we need yet another modification of the ERG differential equations. This is the subject of the next section.

## 5 Third rewriting: Wilson-Fisher fixed point

Assuming self-similarity, we write our  $2n$ -point vertex as

$$\mathcal{V}_{2n}(p_1, \dots, p_{2n}; m^2, \lambda)$$

By introducing counterterms for the squared mass and wave function, we wish to modify the differential equations so that the following conditions are met:

$$\mathcal{V}_2(0, 0; 0, \lambda) = a_2 \lambda \quad (62)$$

$$\left. \frac{\partial}{\partial m^2} \mathcal{V}_2(0, 0; m^2, \lambda) \right|_{m^2=0} = 0 \quad (63)$$

$$\left. \frac{\partial}{\partial p^2} \mathcal{V}_2(p, -p; 0, \lambda) \right|_{p^2=0} = 0 \quad (64)$$

Note that these are not asymptotic conditions. The first and second conditions determines the mass counterterm, and the third the wave function renormalization. We define  $\lambda$  by

$$\mathcal{V}_4(0, 0, 0, 0; 0, \lambda) = -\lambda \quad (65)$$

The modified ERG differential equations are given as follows:

$$\begin{aligned} & \frac{\partial}{\partial t} (e^{-2t} \mathcal{V}_2(pe^t, -pe^t; m^2(t), \lambda(t))) \\ &= e^{-2t} \{ \beta_m(\lambda(t)) m^2(t) + \eta(\lambda(t)) (p^2 e^{2t} + m^2(t)) \} \\ & \quad + s(pe^t; m^2(t), \lambda(t)) \cdot e^{-2t} \mathcal{V}_2(pe^t, -pe^t; m^2(t), \lambda(t)) \\ & \quad + \{ e^{-2t} \mathcal{V}_2(pe^t, -pe^t; m^2(t), \lambda(t)) \}^2 \frac{\Delta(pe^t)}{p^2 + e^{-2t} m^2(t)} \\ & \quad + \frac{1}{2} \int_q \frac{\Delta(qe^t)}{q^2 + e^{-2t} m^2(t)} e^{-t} \mathcal{V}_4(qe^t, -qe^t, pe^t, -pe^t; m^2(t), \lambda(t)), \quad (66) \\ & \frac{\partial}{\partial t} (e^{-y_{2n}t} \mathcal{V}_{2n}(p_1 e^t, \dots, p_{2n} e^t; m^2(t), \lambda(t))) \\ &= \frac{1}{2} \sum_{i=1}^{2n} s(p_i e^t; m^2(t), \lambda(t)) \cdot e^{-y_{2n}t} \mathcal{V}_{2n}(p_1 e^t, \dots, p_{2n} e^t; m^2(t), \lambda(t)) \end{aligned}$$

$$+ \sum_{\text{partitions}} \left( \text{diagram with two circles connected by a thick red line} \right) + \frac{1}{2} \int_q \text{diagram with a circle and a red self-loop}$$

where the thick line with momentum  $q$  denotes

$$\frac{\Delta(qe^t)}{q^2 + e^{-2t} m^2(t)}$$

In the above,  $\beta_m(\lambda)$  is the anomalous dimension of  $m^2$  so that

$$\frac{d}{dt}m^2(t) = (2 + \beta_m(\lambda(t)))m^2(t) + c(\lambda(t)) \quad (67)$$

and  $\frac{1}{2}\eta(\lambda)$  is the anomalous dimension of the scalar field. The function  $s$  is defined by

$$s(p; m^2, \lambda) \equiv -\eta(\lambda)K(p) + (\beta_m(\lambda)m^2 + c(\lambda)) \frac{1 - K(p)}{p^2 + m^2} \quad (68)$$

The beta function  $\beta(\lambda)$ , defined as usual by

$$\frac{d}{dt}\lambda(t) = \beta(\lambda(t)), \quad (69)$$

and the other three functions  $\beta_m(\lambda)$ ,  $c(\lambda)$ ,  $\eta(\lambda)$  are determined so that the above ERG differential equations satisfy the four conditions (62-65). Using the notation

$$A_{2n}(p_1, \dots, p_{2n}; \lambda) \equiv \mathcal{V}_{2n}(p_1, \dots, p_{2n}; 0, \lambda) \quad (70)$$

$$B_{2n}(p_1, \dots, p_{2n}; \lambda) \equiv \left. \frac{\partial}{\partial m^2} \mathcal{V}_{2n}(p_1, \dots, p_{2n}; m^2, \lambda) \right|_{m^2=0} \quad (71)$$

$$C_{2n}(p_1, \dots, p_{2n}; \lambda) \equiv \left. \frac{\partial^2}{(\partial m^2)^2} \mathcal{V}_{2n}(p_1, \dots, p_{2n}; m^2, \lambda) \right|_{m^2=0} \quad (72)$$

we find

$$a_2(\beta(\lambda) - \lambda + \lambda\eta(\lambda)) - c(\lambda) = \frac{1}{2} \int_q \frac{\Delta(q)}{q^2} (A_4(q, -q, 0, 0; \lambda) + \lambda) \quad (73)$$

$$\begin{aligned} & \beta_m(\lambda) + \eta(\lambda) - c(\lambda)C_2(0, 0; \lambda) \\ &= \frac{1}{2} \int_q \Delta(q) \left( -\frac{B_4(q, -q, 0, 0; \lambda)}{q^2} + \frac{A_4(q, -q, 0, 0; \lambda)}{q^4} \right) \end{aligned} \quad (74)$$

$$\begin{aligned} & \eta(\lambda) - c(\lambda) \frac{\partial}{\partial p^2} B_2(p, -p; \lambda) \Big|_{p^2=0} \\ &= -\frac{1}{2} \frac{\partial}{\partial p^2} \int_q \frac{\Delta(q)}{q^2} A_4(q, -q, p, -p; \lambda) \Big|_{p^2=0} \end{aligned} \quad (75)$$

$$\begin{aligned} & \beta(\lambda) - \lambda + 2\lambda\eta(\lambda) - c(\lambda)B_4(0, 0, 0, 0; \lambda) \\ &= -\frac{1}{2} \int_q \frac{\Delta(q)}{q^2} A_6(q, -q, 0, 0, 0, 0; \lambda) \end{aligned} \quad (76)$$

These imply

$$\beta(\lambda) - \lambda = O(\lambda^2) \quad (77)$$

$$\beta_m(\lambda) = O(\lambda) \quad (78)$$

$$c(\lambda) = O(\lambda^2) \quad (79)$$

$$\eta(\lambda) = O(\lambda^2) \quad (80)$$

and we can introduce the following series expansions:

$$\beta(\lambda) - \lambda = \beta_1\lambda^2 + \beta_2\lambda^3 + \dots \quad (81)$$

$$\beta_m(\lambda) = \beta_{m1}\lambda + \beta_{m2}\lambda^2 + \dots \quad (82)$$

$$c(\lambda) = c_2\lambda^2 + \dots \quad (83)$$

$$\eta(\lambda) = \eta_2\lambda^2 + \dots \quad (84)$$

The Wilson-Fisher point  $(m^{2*}, \lambda^*)$  is found from

$$\beta(\lambda^*) = 0 \quad (85)$$

$$(2 + \beta_m(\lambda^*))m^{2*} + c(\lambda^*) = 0 \quad (86)$$

where  $\beta_m^* \equiv \beta_m(\lambda^*)$  and  $\eta^* \equiv \eta(\lambda^*)$  are the anomalous dimensions of the squared mass and scalar field, respectively. Both should be independent of the choice of the cutoff function  $K$ , but we have not been able to demonstrate the independence. If we truncate the series expansions in  $\lambda$ , both  $\beta_m^*$  and  $\eta^*$  depend on the choice of  $K$  as we will see later.

## 5.1 Integral equations

For perturbative calculations in powers of  $\lambda$ , we have found it convenient to convert the ERG differential equations and the four conditions (62 - 65) into integral equations. For the two-point vertex, we obtain

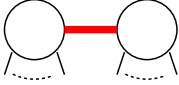
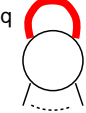
$$\begin{aligned} & e^{-2t} \mathcal{V}_2(pe^t, -pe^t; m^2(t), \lambda(t)) \\ = & \int_{-\infty}^t dt' \left[ \beta_m(\lambda(t')) e^{-2t'} m^2(t') + \eta(\lambda(t')) (p^2 + e^{-2t'} m^2(t')) \right. \\ & + s(pe^{t'}; m^2(t'), \lambda(t')) e^{-2t'} \mathcal{V}_2(pe^{t'}, -pe^{t'}; m^2(t'), \lambda(t')) \\ & + \eta(\lambda(t')) a_2 e^{-2t'} \lambda(t') \\ & + \left( e^{-2t'} \mathcal{V}_2(pe^{t'}, -pe^{t'}; m^2(t'), \lambda(t')) \right)^2 \frac{\Delta(pe^{t'})}{p^2 + e^{-2t'} m^2(t')} \\ & + \frac{1}{2} \int_q \Delta(qe^{t'}) \left( \frac{e^{-t'} \mathcal{V}_4(qe^{t'}, -qe^{t'}, pe^{t'}, -pe^{t'}; m^2(t'), \lambda(t'))}{q^2 + e^{-2t'} m^2(t')} \right. \\ & \quad \left. \left. - \frac{e^{-t'} A_4(qe^{t'}, -qe^{t'}, 0, 0; \lambda(t'))}{q^2} \right) \right] + e^{-2t} a_2 \lambda(t) \end{aligned} \quad (87)$$

For the four-point vertex, we obtain

$$\begin{aligned} & e^{-t} \mathcal{V}_4(p_1 e^t, \dots, p_4 e^t; m^2(t), \lambda(t)) \\ = & \int_{-\infty}^t dt' \left[ \frac{1}{2} \sum_{i=1}^4 s(p_i e^{t'}; m^2(t'), \lambda(t')) \cdot e^{-t'} \mathcal{V}_4(p_1 e^{t'}, \dots, p_4 e^{t'}; m^2(t'), \lambda(t')) \right. \\ & - 2\eta(\lambda(t')) e^{-t'} \lambda(t') + c(\lambda(t')) e^{-t'} B_4(0, 0, 0, 0; \lambda(t')) \\ & + \sum_{i=1}^4 e^{-2t'} \mathcal{V}_2(p_i e^{t'}, -p_i e^{t'}; m^2(t'), \lambda(t')) \frac{\Delta(p_i e^{t'})}{p_i^2 + e^{-2t'} m^2(t')} \\ & \quad \times e^{-t'} \mathcal{V}_4(p_1 e^{t'}, \dots, p_4 e^{t'}; m^2(t'), \lambda(t')) \\ & + \frac{1}{2} \int_q \Delta(qe^{t'}) \left\{ \frac{\mathcal{V}_6(qe^{t'}, -qe^{t'}, p_1 e^{t'}, \dots, p_4 e^{t'}; m^2(t'), \lambda(t'))}{q^2 + e^{-2t'} m^2(t')} \right. \\ & \quad \left. \left. - \frac{A_6(qe^{t'}, -qe^{t'}, 0, 0, 0, 0; \lambda(t'))}{q^2} \right\} \right] - e^{-t} \lambda(t) \end{aligned} \quad (88)$$

For the six- and higher-point vertices, we obtain

$$\begin{aligned}
e^{-y_{2n}t} \mathcal{V}_{2n \geq 6}(p_1 e^t, \dots, p_{2n} e^t; m^2(t), \lambda(t)) &= \int_{-\infty}^t dt' \left[ \right. \\
&\frac{1}{2} \sum_{i=1}^{2n} s(p_i e^{t'}; m^2(t'), \lambda(t')) \cdot e^{-y_{2n}t'} \mathcal{V}_{2n}(p_1 e^{t'}, \dots, p_{2n} e^{t'}; m^2(t'), \lambda(t')) \\
&+ \sum_{\text{partitions}} \text{Diagram 1} + \frac{1}{2} \int_q \text{Diagram 2} \left. \right]
\end{aligned}$$

where the thick line with momentum  $q$  denotes

$$\frac{\Delta(qe^{t'})}{q^2 + e^{-2t'} m^2(t')}$$

It is straightforward to check that the above integral equations give the correct  $t$ -dependence and satisfy the conditions (62 - 65).

## 5.2 Results of perturbative calculations

We will not give any details of the perturbative calculations, and write down only the relevant results. (See Appendix A for some details.)

Using the series expansions of  $\beta(\lambda)$ , we find the fixed point

$$\lambda^* \simeq \frac{1}{-\beta_1} \tag{89}$$

At lowest non-trivial order, the anomalous dimensions are obtained as

$$\beta_m^* \simeq \frac{\beta_{m1}}{-\beta_1} \tag{90}$$

$$\eta^* \simeq \frac{\eta_2}{\beta_1^2} \tag{91}$$

All the coefficients are given in terms of the cutoff function  $K$ . For example, if we choose

$$K(q) \equiv \begin{cases} 1 & \text{for } q^2 < 1 \\ \frac{a^2 - q^2}{a^2 - 1} & \text{for } 1 < q^2 < a^2 \\ 0 & \text{for } q^2 > a^2 \end{cases} \tag{92}$$

(Fig. 4) then we obtain

$$\beta_1 = -3 \int_q \frac{\Delta(q)(1 - K(q))}{q^4} = -\frac{1}{\pi^2} \frac{a+2}{(a+1)^2} \tag{93}$$

$$\beta_{m1} = -\frac{1}{2} \int_q \frac{\Delta(q)}{q^4} = -\frac{1}{\pi^2} \frac{1}{2(a+1)} \tag{94}$$

so that the one-loop result

$$\beta_m^* \simeq \frac{\beta_{m1}}{-\beta_1} = -\frac{1+a}{2(2+a)} \tag{95}$$

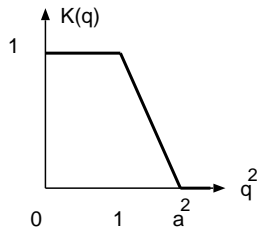


Figure 4: A choice for the cutoff function.

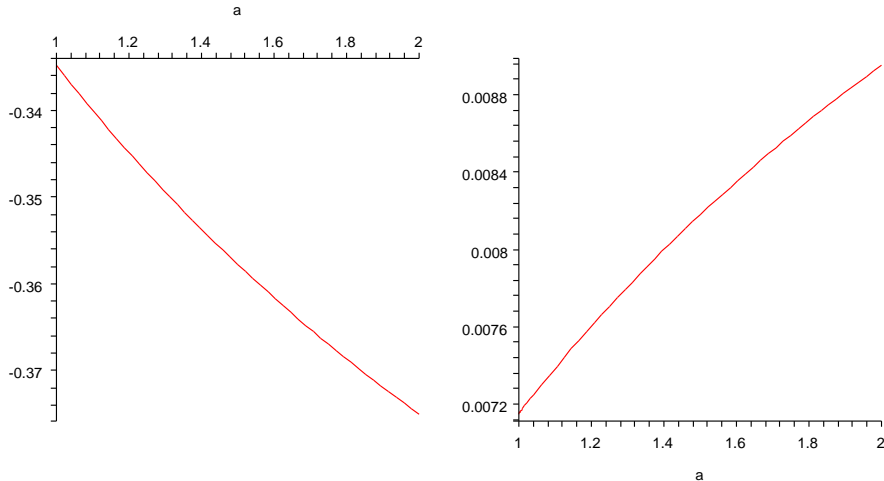


Figure 5:  $\beta_m^*$  and  $\eta^*$  for  $1 < a < 2$ .

is obtained. This is between  $-\frac{1}{3}$  ( $a = 1$ ) and  $-\frac{1}{2}$  ( $a \rightarrow \infty$ ), comparable to the best fit  $-0.41$  to various experimental results. (For example, see Table 5.4.2 of [5].) With the above choice for  $K$ , we plot the  $a$  dependence of the critical exponents  $\beta_m^*$  at one-loop and  $\eta^*$  at two-loop (Fig. 5). Our value for  $\eta^*$  turns out to be too small compared to the experimental fit  $0.03 - 0.06$ .

In the limit  $a \rightarrow 1+$ , our formalism is expected to be equivalent to the Wegner-Houghton ERG equations.[6] But we have not yet examined this expected equivalence.

## 6 Conclusion

In this paper we have carefully examined the nature of the solutions to Polchinski's ERG differential equations. We have shown how the ordinary RG equations of renormalized parameters arise from ERG. We have also shown the necessity to modify the ERG differential equations, first for self-similarity, and second for the Wilson-Fisher fixed point. In all this analysis, we have found it helpful to use the integral equation approach.

Our perturbative calculations of the critical exponents are reminiscent of Parisi's use of the Callan-Symanzik equations to do the same.[7] One undesirable

feature common in both is the absence of an obvious expansion parameter for the critical exponents. Notice that  $\lambda^*$  given by Eq. (89) is not necessarily a small number. Introducing  $N \gg 1$  number of fields is an easy way to rectify the problem, but it evades the question of validity of perturbation theory presented in this paper.

The perturbative method given in this paper is by no means the only way to calculate the critical exponents of the Wilson-Fisher fixed point. Typically the calculations are done non-perturbatively after truncating the ERG equations. See, for example, ref. [8] and references therein.

## A Results of perturbative expansions

Using the notation

$$A_{2n}(p_1, \dots, p_{2n}; \lambda) = \sum_{k=1}^{\infty} \lambda^k A_4^{(k)}(p_1, \dots, p_{2n}) \quad (96)$$

we obtain the following results:

$$A_4^{(1)}(q, -q, 0, 0) = -1 \quad (97)$$

$$A_6^{(2)}(q, -q, 0, 0) = 6 \frac{1 - K(q)}{q^2} \quad (98)$$

at one-loop, and

$$A_4^{(2)}(q, -q, 0, 0) = -2a_2 \frac{1 - K(q)}{q^2} + \int_r \left( \frac{1 - K(r)}{r^2} \frac{1 - K(q+r)}{(q+r)^2} - \frac{(1 - K(r))^2}{r^4} \right) \quad (99)$$

$$B_4^{(2)}(q, -q, 0, 0) = \beta_{m1} \frac{\int_{-\infty}^0 dt e^t (1 - K(qe^t))}{q^2} + 2a_2 \frac{1 - K(q)}{q^4} - \int_r \frac{(1 - K(r))^2}{r^6} - 2 \int_r \frac{1 - K(r)}{r^4} \frac{1 - K(q+r)}{(q+r)^2} \quad (100)$$

$$\left. \frac{\partial}{\partial p^2} A_4^{(2)}(q, -q, p, -p) \right|_{p^2=0} = \left. \frac{\partial}{\partial p^2} \int_r \frac{1 - K(r)}{r^2} \frac{1 - K(p+q+r)}{(p+q+r)^2} \right|_{p^2=0} \quad (101)$$

$$A_6^{(3)}(q, -q, 0, 0, 0, 0) = 18a_2 \frac{(1 - K(q))^2}{q^4} - 8\beta_1 \frac{1 - K(q)}{q^2} - 3 \int_r \frac{\Delta(r)(1 - K(r))^2}{r^6} - 12 \frac{1 - K(q)}{q^2} \int_r \frac{1 - K(r)}{r^2} \frac{1 - K(q+r)}{(q+r)^2} - 12 \int_r \frac{(1 - K(r))^2 (1 - K(q+r))}{r^4 (q+r)^2} \quad (102)$$

at two-loop.

Now, from Eqs. (73-76) we obtain

$$\beta_{m1} = \frac{1}{2} \int_q \frac{\Delta(q)}{q^4} A_4^{(1)}(q, -q, 0, 0) = -\frac{1}{2} \int_q \frac{\Delta(q)}{q^4} \quad (103)$$



$$\beta_1 = -\frac{1}{2} \int_q \frac{\Delta(q)}{q^2} A_6^{(2)}(q, -q, 0, 0, 0, 0) = -3 \int_q \frac{\Delta(q)(1-K(q))}{q^4} \quad (104)$$

at one-loop, and

$$a_2 \beta_1 - c_2 = \frac{1}{2} \int_q \frac{\Delta(q)}{q^2} A_4^{(2)}(q, -q, 0, 0) \quad (105)$$

$$\beta_{m2} + \eta_2 = \frac{1}{2} \int_q \Delta(q) \left( -\frac{B_4^{(2)}(q, -q, 0, 0)}{q^2} + \frac{A_4^{(2)}(q, -q, 0, 0)}{q^4} \right) \quad (106)$$

$$\eta_2 = -\frac{1}{2} \frac{\partial}{\partial p^2} \int_q \frac{\Delta(q)}{q^2} A_4^{(2)}(q, -q, p, -p) \Big|_{p^2=0} \quad (107)$$

$$\beta_2 + 2\eta_2 = -\frac{1}{2} \int_q \frac{\Delta(q)}{q^2} A_6^{(3)}(q, -q, 0, 0, 0, 0) \quad (108)$$

at two-loop, where

$$a_2 = \frac{1}{2} \int_q \frac{\Delta(q)}{q^2} \quad (109)$$

Hence, we get

$$c_2 = -\frac{1}{2} \int_{q,r} \frac{1-K(q)}{q^2} \frac{1-K(r)}{r^2} \frac{\Delta(q+r)}{(q+r)^2} = -\frac{1}{6} \frac{1}{(4\pi)^2} \quad (110)$$

$$\eta_2 = -\frac{1}{2} \frac{\partial}{\partial p^2} \int_{q,r} \frac{\Delta(q)}{q^2} \frac{1-K(r)}{r^2} \frac{1-K(p+q+r)}{(p+q+r)^2} \Big|_{p^2=0} \quad (111)$$

Note that  $c_2$  is the same as  $C$  given at the end of sect. 2 and independent of the choice of  $K$ . We can also obtain  $\beta_2, \beta_{m2}$  from (106, 108) by using the calculated four-point vertices. In the main text we have quoted the results for the critical exponents using a particular cutoff function  $K$  with one parameter  $1 < a < 2$ .

## References

- [1] K. G. Wilson and John B. Kogut. The renormalization group and the epsilon expansion. *Phys. Rept.*, 12:75–200, 1974.
- [2] Joseph Polchinski. Renormalization and effective lagrangians. *Nucl. Phys.*, B231:269–295, 1984.
- [3] Hidenori Sonoda. Bootstrapping perturbative perfect actions. *Phys. Rev.*, D67:065011, 2003.
- [4] Hidenori Sonoda. Beta functions in the integral equation approach to the exact renormalization group. 2003.
- [5] P. M. Chaikin and T. C. Lubensky. *Principles of condensed matter physics*. Cambridge, 1995.
- [6] Franz J. Wegner and Anthony Houghton. Renormalization group equation for critical phenomena. *Phys. Rev.*, A8:401–412, 1973.
- [7] G. Parisi. *Statistical field theory*. Perseus Books Group, 1998.
- [8] J. P. Blaizot, Ramon Mendez Galain, and Nicolas Wschebor. A new method to solve the non perturbative renormalization group equations. 2005.